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1. The establishment and study of invariant quantities, i.e. quantities, undergoing relatively little change during motion with some velocity, has for a long time occupied a large place in meteorological investigations. It follows to recognize this as quite natural, since the basic properties of a process on the very large scale are characterized by those (factors), namely what quantities appear to be invariant and what is the advection velocity of these invariant quantities.

It is well known how great the practical significance of "conservative" quantities (which appear, evidently, as one of the forms of the invariant quantities) is in the analysis of atmospheric processes. It is sufficient to point to the application of potential and equivalent-potential temperature in thermodynamics of the atmosphere, and to numerous applications of Bernoulli's equation, the purpose of which consists in the determination of a conservative quantity for established processes, taking place in an incompressible ideal fluid or in a barotropic ideal medium.

It is known that the quantity

$$B = \frac{V^2}{2} + gz + \int \frac{dp}{\rho(p)},$$

where $\vec{V}$ is the velocity, g is the acceleration of gravity, p is pressure and ρ is the density of the air, does not remain strictly constant along the lines of flow (just by virtue of the presence of viscosity, not to mention other effects). Nevertheless, the application of the Bernoulli equation together with the condition of conservation of potential temperature

permits us to solve, quite sufficiently for practical accuracy, a great number of aerodynamic, technical and meteorological problems. To find a dynamic characteristic, which is conserved during non-stationary atmospheric processes with approximately the same degree of accuracy as the quantity β during established processes, means it would considerably facilitate many meteorological investigations. However, unti recently, almost all the accomplished transformations of the dynamic equations of the atmosphere for the purpose of the establishment of invariant quantities has been based on serious simplifications of the actual relatic between meteorological quantities. As a result the properties of the invariance of such quantities ~~is~~ ARE realized only highly approximately.

The work of Charney [8], who approached quite close to the determina of an invariant, the expression for which is given later [formula (13)], i of significant interest. However, as far as a suitable conclusion realize with the neglect of terms of the equations of the first order of smallness that relation (13) defines a quantity, invariant only as a first approxima

We shall see shortly, that quantities of the first order of smallnes usually constitute 15-20% of the main terms of the equatinns, so that the neglect of several such terms can lead to relatively significant errors. the present work special transformations of the equations are realized, whi permit us to obtain an invariant of the second approximation, i.e. limit ourselves to neglect of terms of the second order of smallness. But first it is necessary to make the concept of large scale atmospheric processes more precise and to examine the basic characteristics of their relation.

2. First of all, let us note, that for the classification of atmospheric motions the criteria of Reynolds and Froude (which are usual

employed for this purpose in hydromechanics) prove to be useless. For all atmospheric motions studied by dynamic meteorology, the Reynolds number is very great and on the strength of this it appears possible to neglect the molecular viscosity (with the exception of the air film, of thickness of the order of a millimeter, which directly borders the underlying surface. The Froude number, in almost all the motions considered in dynamic meteorology (excepting hurricanes, tornadoes, etc., and also motion in cumulus clouds) is very small. Therefore, practically, it is possible to consider the vertical component of the inertia force as a small addition to the force of gravity, while in a majority of cases these additions are simply neglected. For that reason, in the nature of a fundamental criteria for the classification of atmospheric motions, it is suitable to use the ratio of the horizontal components relative to the Coriolis acceleration of the air particles.

Introducing the designations:

U is the horizontal velocity of the air motion relative to the earth, L is the scale of the motion, $\ell = 2\omega \sin \phi$, where ω is the angular velocity of the earth's rotation, ϕ is latitude, De is the ratio of the above mentioned accelerations, we obtain

(1) $$De = \frac{U^2}{L} : \ell U = \frac{U}{\ell L}$$

We shall call the motion, in which the relative acceleration appears to be determined for $\left(L \ll \frac{U}{\ell} \right)$, small-scale; motion, in which the Coriolis acceleration appears to be determined for $\left(L \gg \frac{U}{\ell} \right)$, large scale and, finally, motion, in which both the accelerations considered

have the same order, medium-scale. For a quantitative estimation of the characteristic scales of the three classes of motion mentioned, it is necessary to remember, that the quantity U has been related to L [5].

Carrying out the corresponding estimations, it is possible to convinc yourself, that outside equatorial regions, motions of scales up to 100 mete inclusively, as a rule, are regarded as small-scale; motions of scales of kilometers and tens of kilometers (in particular, breezes) -- medium-scale and hundreds and thousands of kilometers -- large-scale.

Considering, that $\frac{U}{L}$ represents the characteristic value of the vertical component of relative vorticity Ω, it is still possible to write down the criteria for large-scale processes in such a form

(2) $$\frac{\Omega}{\ell} << 1$$

Let us note that ℓ represents the vertical component of the vorticity of the fundamental motion (vorticity of the earth's rotation(?) - Trans.). Therefore large-scale motion is characterized by that condition, that the vertical component of the vorticity of the fundamental motion of particles many times exceeds the vertical component of the relative vortici

Condition (1) is often used for the simplification of the dynamic equations of the atmosphere, beginning with the known work of Kibel [3]. Here in the Fridman equation for the change of vorticity terms of the firs order of smallness are neglected (i.e. quantities, the ratio of which to main terms of the equation have the order $\frac{\Omega}{\ell}$) and correspondingly in the equations of horizontal motion quantities of the second order of smallness. For such simplifications it proves to be possible to obtain o the invariant of the first approximation, so that we shall have to carry o

a more detailed analysis of the smallness of the various quantities.

3. In the work [6], an estimation of the terms of the equations of horizontal motion and heat flow was made, and also considered were the two equations, obtained by the application of the vorticity and divergence operation to the vector equation for horizontal acceleration. From carry[illegible] out the analysis further the following conclusions are of interest.

(a) It is possible to take the characteristic value of the ratio equal to 0.2 for macro-scale motion for the characteristic length L the order of 300-500 km. Further, quantities, having the order $\frac{\Omega}{\ell}$ we shall designate through ε.

(b) The ratio of the divergence of the horizontal velocity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ to ℓ produces a value of second order of smallness, i.e.

$$O\left[\frac{1}{\ell}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] = \varepsilon^2$$

$(O[f]$ — order of the quantity $f)$

(c) The ratio of the quantities

$$w\frac{\partial V_s}{\partial z} : V_s\frac{\partial V_s}{\partial s}$$

where V_s is the horizontal component of velocity $(u$ or $v)$, s is the horizontal coordinate $(x$ or $y)$, z is height, w, is vertical velocity, is a quantity the order of ε. However the quantities $w\frac{\partial T}{\partial z}$ and $V_s\frac{\partial T}{\partial s}$ have the same order in the free atmosphere.

(d) Neglect of turbulent friction for the macro-scale motions stu[illegible] in the free atmosphere seems to be in general sufficiently justified.

Proceeding from the last conclusion, in the future let us consider the equations of an ideal fluid. Let us also regard the motion in the free atmosphere as adiabatic, leaning on the fact of the smallness of heat flow, shown by C. C. Gaigerov and V. G. Kastrov [1], [2] on the basis of data of special free-balloon investigations. Thus, the invariance of potential temperature is assumed from the very beginning. Let us take advantage of the dynamic equations of the atmosphere in the variables t (time), x (the x axis is directed toward the east), y (the y axis is directed toward the north), $\varsigma = \frac{p}{p_0}$ (the ratio of the pressure to its standard value $p_0 = 1000$ mb). The horizontal componen of velocity u, v, the geopotential $\phi = \int g\,dz$, the potential temperature θ and the quantity $\widetilde{w}$, equal to

(3) $$\widetilde{w} = -\frac{g\rho}{p_0} w + \frac{\rho}{p_0}\left(\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y}\right)$$

appear as the unknown functions.

Here w is the veritcal velocity and ρ is the density of the air. The quantity $\widetilde{w}$ is proportional to the individual change of pressure $\left(\widetilde{w} = \frac{1}{p_0}\frac{dp}{dt}\right)$.

The system of equations which interests us have been encountered often recently in investigations by dynamic meteorology (see, for example, [7], where it was also indicated by way of a conclusion). Let us copy it.

The equations which express the horizontal acceleration,

(4) $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \widetilde{w}\frac{\partial u}{\partial \varsigma} - \ell v = -\frac{\partial \phi}{\partial x}$$

(5) $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \widetilde{w}\frac{\partial v}{\partial \varsigma} + \ell u = -\frac{\partial \phi}{\partial y}$$

The equation of continuity

(6) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \widetilde{w}}{\partial \zeta} = 0$$

The equation of the adiabatic process

(7) $$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + \widetilde{w}\frac{\partial \theta}{\partial \zeta} = 0$$

Finally, the relation between θ and the geopotential is found from the hydrostatic equation and the equation of state

(8) $$\theta = -\frac{1}{R}\zeta^{1-\lambda}\frac{\partial \phi}{\partial \zeta} \quad (\lambda = 0.285)$$

In connection with this, that the angle between the directed normal to the isobaric surface and the vertical is very small, then the quantities $\frac{\partial u}{\partial t}$, $u\frac{\partial u}{\partial x}$, etc., which appear in equations (4)-(6), differ qu little from those same values, determined in the system of variables t, x The relative difference between, for example, the values of $\frac{\partial u}{\partial x}$, calculated under constant ζ or under constant z usually do not exceed 3 - 5%. However the presence of these small differences (and also of course, that $\partial \widetilde{w}/\partial \zeta$ differs from $\frac{\partial w}{\partial z}$) leads to appreciable simplification of the form of the continuity equation, from which the small quantity $\frac{1}{\rho}\frac{d\rho}{dt}$ is eliminated.

It follows to note, that changes, under the indicated substitution of variable values of the derivatives with respect to time and the horizo direction, of the potential temperature are significantly larger than the corresponding values of the velocity components, and could reach 10-20%. This fact is found in direct connection with the conclusion, mentioned

above, concerning the variability of the temperature (also the potential temperature) and the wind in various directions.

4. Let us now construct a table, analogous to [6], of the orders of the quantities, appearing in equations (4)-(8). For this we can profit partially by the data of the work of [6], introducing the corresponding conversion factors to the new variables. Some new results were obtained by means of the treatment of actual data. Tables 1 and 1a contain the mean square deviations from the standard derivatives (divided differences) of the quantities $u, v, \phi, \theta, \tilde{w}$ and $\Omega_\rho = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ (derivatives determined for constant ρ) for the layer from 0.5 up to 6-7 km. The space interval for the determination of the horizontal differences of meteorological quantities was chosen equal to $\Delta s = 500$ k, the timey interval equal to $\Delta t =$ 12 hrs. and the interval throug the variable ρ: $\Delta \rho = 0.25$.

In numerous cases the norm (the climatic value, averaged with respect to time and space) of the derivative proved to be significantly less, than its standard. However, the systematic increase of the geopotential and potential temperature with the drop in pressure leads, for large scale motion, to that conclusion, that the norms of the derivatives of these elements with respect to ρ significantly surpass the standards of those same quantities. It is clear, that in the cases mentioned, the characteristic meaning of the quantity appears to be its norm. The corresponding graphs of the tables contain two quantities: upward the standard, below the norm. It follows to keep in mind that, owing to the geographic variability of the derivatives of meteorological elements, the mean value of the derivative, computed in a definite region and for a definite level (layer), can differ appreciably from the values, indicated

TABLE I

andard derivatives (divided differences) of the horizontal velocity, geopotential

and potential temperature

$\frac{\partial}{\partial s}$	$\frac{\partial}{\partial t}$	$\frac{\partial}{\partial \zeta}$	$\frac{\partial^2}{\partial \zeta^2}$	$\frac{\partial^2}{\partial s\,\partial t}$	$\frac{\partial^2}{\partial s\,\partial \zeta}$	$\frac{\partial^2}{\partial t^2}$	$\frac{\partial^2}{\partial t\,\partial \zeta}$	$\frac{\partial^2}{\partial \zeta^2}$
$.2\times10^{-5}$	1.2×10^{-4}	2×10^{11}	3×10^{11}	3×10^{-10}	5×10^{-5}	5×10^{-9}	5×10^{-4}	10
1×10^{-3}	1×10^{-2}	$\frac{10^4}{1\times10^5}$	2×10^{-9}	3×10^{-3}	3×10^{-3}	2×10^{-10}	3×10^{-2}	$\frac{10^4}{1\times1}$
6×10^{-6}	8×10^{-5}	$\frac{10^1}{5\times10^1}$	2×10^{-11}	2×10^{-10}	3×10^{-5}	3×10^{-9}	3×10^{-4}	10^2

TABLE Ia

:andard value of $\tilde{w}$, of the component of vorticity Ω_ζ and of their derivatives

(divided differences)

	STANDARD VALUE	$\frac{\partial}{\partial s}$	$\frac{\partial}{\partial t}$	$\frac{\partial}{\partial \zeta}$
$\tilde{w}$	10^{-6}	2×10^{-12}	2×10^{-11}	2.5×10^{-6}
Ω_ζ	2×10^{-5}	4×10^{-11}	4×10^{-10}	7×10^{-5}

in the table. Nevertheless, from our data, the relative deviations of the local values of the derivative from its tabular value, as a rule, do not exceed 30%. One can expect the greater discrepancies (from one-half up to double the tabular values) there, where the numerical factor 10 was not indicated in the power.

5. Let us apply the vorticity operation to the equation for horizontal acceleration; for that let us differentiate equation (5) with respect to x and let us subtract from it (4) differentiated with respect to y. After elementary calculations we find

$$(9)\quad \underline{\frac{\partial \Omega_\rho}{\partial t} + u\frac{\partial \Omega_\rho}{\partial x} + v\frac{\partial(\Omega_\rho + l)}{\partial y}} + \tilde{w}\frac{\partial \Omega_\rho}{\partial \rho} + \Omega_\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial \tilde{w}}{\partial x}\frac{\partial v}{\partial \rho} - \frac{\partial \tilde{w}}{\partial y}\frac{\partial u}{\partial \rho} + \underline{l\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} = 0$$

In equation (9) the chief terms are underlined by the thick straight line and the terms of first order of smallness relative to the chief terms are underlined by the thin wavy line. Here the underlined conclusions calculated above are a and b. However, an estimation of all the values could be obtained directly from the tables, if we consider, that by (6)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial \tilde{w}}{\partial \rho}$$

From equation (9) it is clear, that due to the presence of the divergence of the horizontal velocity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, the quantity $l + \Omega_\rho$ does not appear to be an invariant even in the first approximation.[1/]

1/ It is obvious, that the first four terms of equation (9) could be presented in the form $\frac{d}{dt}(l + \Omega_\rho)$, in so far as l does not depend on t, x, ρ.

The attempt of Rossby to introduce the special invariant quantity -- "the potential vorticity" is justified, as A. M. Obukhov [4] mentions, only for serious simplifying assumptions. The method of transition to an invariant quantity consists in the application of equation (9) at such a level, where the divergence of the horizontal velocity is very small and could be neglected, which is also another assumption given for the first time by Rossby.

We can show some difference of the standard values of $\frac{\partial \Omega_\ell}{\partial t}$ and $\ell \frac{\partial \tilde{w}}{\partial \zeta}$ even by the mean data, given in table 1a. Actually, the standard value of $\frac{\partial \Omega_\ell}{\partial t}$ is 4×10^{-10}, and the value of $\ell \frac{\partial \tilde{w}}{\partial \zeta}$ is 3×10^{-10} (for $\ell = 1.2 \times 10^{-4}$, which corresponds to latitude 55°).

If we look for the layer, in which the ratio $\ell \frac{\partial \tilde{w}}{\partial \zeta} : \frac{\partial \Omega_\ell}{\partial t}$ would be equal to 0.5 in the mean, then the quantity $\ell + \Omega_\zeta$ in such a layer appears to be invariant with an accuracy to $\varepsilon^{1/2}$. Let us note, that the so called "barotropic scheme" of the precalculation of pressure depends on the use of this quite approximate invariant.

Now let us introduce the invariant of the first approximation from (9) and (7). For this let us neglect in (9) terms of the first order of smallness $\frac{\partial \tilde{w}}{\partial x}\frac{\partial v}{\partial \zeta} - \frac{\partial \tilde{w}}{\partial y}\frac{\partial u}{\partial \zeta}$, let us replace $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ by $-\frac{\partial \tilde{w}}{\partial \zeta}$, and everywhere in the remaining terms let us replace the horizontal velocity components by their geostrophic values

$$(10) \qquad u_g = -\frac{1}{\ell}\frac{\partial \phi}{\partial y}, \qquad v_g = \frac{1}{\ell}\frac{\partial \phi}{\partial x}$$

Estimating the order of the quantities in equations (4) and (5), we are satisfied, that

$$\frac{u - u_g}{\sqrt{u_g^2 + v_g^2}} \quad , \quad \frac{v - v_g}{\sqrt{u_g^2 + v_g^2}}$$

are essentially quantities of order ε . Therefore, when deducing the invariant of the first approximation the replacement of the actual wind by the geostrophic wind is completely natural.

So, with accuracy to the first order of smallness

(9a) $$\frac{\partial \Omega_f}{\partial t} + u_g \frac{\partial \Omega_f}{\partial x} + v_g \frac{\partial (\Omega_f + \ell)}{\partial y} + \tilde{w} \frac{\partial \Omega_f}{\partial \int} - (\ell + \Omega_f) \frac{\partial \tilde{w}}{\partial \int} = 0$$

(7a) $$\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + \tilde{w} \frac{\partial \theta}{\partial \int} = 0$$

Let us differentiate equation (7a) with respect to $\int$. On the strength of (10) and (8)

$$\frac{\partial u_g}{\partial \int} \frac{\partial \theta}{\partial x} + \frac{\partial v_g}{\partial \int} \frac{\partial \theta}{\partial y} = -\frac{1}{\ell} \left(-\frac{\partial^2 \phi}{\partial y \partial \int} \frac{\partial \theta}{\partial x} + \frac{\partial^2 \phi}{\partial x \partial \int} \frac{\partial \theta}{\partial y} \right) =$$

$$= \frac{R \int^{\lambda - 1}}{\ell} \left(\frac{\partial \theta}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} \right) = 0$$

Therefore, designating $\partial \theta / \partial \int = \Gamma$, we find

$$\frac{\partial \Gamma}{\partial t} + u_g \frac{\partial \Gamma}{\partial x} + v_g \frac{\partial \Gamma}{\partial y} + \tilde{w} \frac{\partial \Gamma}{\partial \int} + \Gamma \frac{\partial \tilde{w}}{\partial \int} = 0$$

or

(11) $$\frac{\partial (\ln \Gamma)}{\partial t} + u_g \frac{\partial (\ln \Gamma)}{\partial x} + v_g \frac{\partial}{\partial y} (\ln \Gamma) + \tilde{w} \frac{\partial (\ln \Gamma)}{\partial \int} + \frac{\partial \tilde{w}}{\partial \int} = 0$$

(11)

Further, let us divide (9a) by $l + \Omega_f$ and present it in the form

$$(12)\qquad \frac{\partial \ln(l+\Omega_f)}{\partial t} + u_g \frac{\partial \ln(l+\Omega_f)}{\partial x} + v \frac{\partial \ln(l+\Omega_f)}{\partial y} + \tilde{w} \frac{\partial \ln(l+\Omega_f)}{\partial \zeta} - \frac{\partial \tilde{w}}{\partial \zeta} = 0$$

Adding (11) and (12), we find

$$(13)\qquad \frac{\partial[\ln \Gamma + \ln(l+\Omega_f)]}{\partial t} + u_g \frac{\partial}{\partial x}[\ln \Gamma + \ln(l+\Omega_f)] + v_g \frac{\partial}{\partial y}[\ln \Gamma + \ln(l+\Omega_f)] + \tilde{w} \frac{\partial}{\partial \zeta}[\ln \Gamma + \ln(l+\Omega_f)] = 0$$

Thus, the quantity $\ln \Gamma + \ln(l + \Omega_f)$ appear[5] invariant in the first approximation. It is clear, that one can say the same about the quantity $(l + \Omega_f)\dfrac{\partial \theta}{\partial \zeta}$.

6. To obtain the invariant of the second approximation let us introduce the following substitution of variables

$$(14)\qquad \xi = x + \frac{v}{l}, \quad \eta = y - \frac{u}{l}$$

Let us construct completely, for example, the calculations, connected with the substitutions (14) for equation (7). Let $\theta(t, x, y, \zeta)$ in the new variables be presented as a function of $F(t, \xi, \eta, \zeta)$.

Then

(15) $$\frac{\partial\theta}{\partial t}=\frac{\partial F}{\partial t}+\frac{\partial F}{\partial \xi}\frac{1}{l}\frac{\partial v}{\partial t}-\frac{\partial F}{\partial \eta}\frac{1}{l}\frac{\partial u}{\partial t}$$

$$\frac{\partial\theta}{\partial x}=\frac{\partial F}{\partial \xi}+\frac{\partial F}{\partial \xi}\frac{1}{l}\frac{\partial v}{\partial x}-\frac{\partial F}{\partial \eta}\frac{1}{l}\frac{\partial u}{\partial x}$$

$$\frac{\partial\theta}{\partial y}=\frac{\partial F}{\partial \eta}+\frac{\partial F}{\partial \xi}\frac{1}{l}\frac{\partial v}{\partial y}-\frac{\partial F}{\partial \eta}\frac{1}{l}\frac{\partial u}{\partial y}-$$

$$-\frac{\partial F}{\partial \xi}\frac{v}{l^2}\frac{dl}{dy}+\frac{\partial F}{\partial \eta}\frac{u}{l^2}\frac{dl}{dy}$$

$$\frac{\partial\theta}{\partial \zeta}=\frac{\partial F}{\partial \zeta}+\frac{\partial F}{\partial \xi}\frac{1}{l}\frac{\partial u}{\partial \zeta}-\frac{\partial F}{\partial \eta}\frac{1}{l}\frac{\partial u}{\partial \zeta}$$

Hence we obtain that

$$\frac{d\theta}{dt}=\frac{\partial\theta}{\partial t}+u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}+\tilde{w}\frac{\partial\theta}{\partial \zeta}=\frac{\partial F}{\partial t}+\left(u+\frac{1}{l}\frac{dv}{dt}-\underline{\frac{v^2}{l^2}\frac{dl}{dy}}\right)\frac{\partial F}{\partial \xi}+$$

$$+\left(v-\frac{1}{l}\frac{du}{dt}+\underline{\frac{uv}{l^2}\frac{dl}{dy}}\right)\frac{\partial F}{\partial \eta}+\tilde{w}\frac{\partial F}{\partial \zeta}$$

Since the characteristic values of $\frac{dl}{dy}=-\frac{2\omega\cos\phi}{a}$ (a is the radius of the earth) is 1.5×10^{-11} , then the ratios of the quantities, underlined by a wavy line, to the wind velocity present a value of the order $\varepsilon^2-\varepsilon^3$. Neglecting them and using (4), (5), (10), we obtain

$$\frac{d\theta}{dt}=\frac{\partial F}{\partial t}+u_g\frac{\partial F}{\partial \xi}+v_g\frac{\partial F}{\partial \eta}+\tilde{w}\frac{\partial F}{\partial \zeta}$$

or replacing the temporary designation F by θ (but θ has already been expressed as a function of the new variables) we shall have

(16)

$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial t} + u_g \frac{\partial\theta}{\partial\xi} + v_g \frac{\partial\theta}{\partial\eta} + \tilde{w}\frac{\partial\theta}{\partial\zeta}$$

Now let us substitute into equation (9) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial\tilde{w}}{\partial\zeta}$ from the continuity equation

(17) $$\underline{\frac{d}{dt}(\ell+\Omega_\zeta) - \ell\frac{\partial\tilde{w}}{\partial\zeta}} - \Omega_\zeta\frac{\partial\tilde{w}}{\partial\zeta} + \frac{\partial\tilde{w}}{\partial x}\frac{\partial v}{\partial\zeta} - \frac{\partial\tilde{w}}{\partial y}\frac{\partial u}{\partial\zeta} = 0$$

As before small terms of the first order have been underlined by a wavy line.

Analogous ~~ly~~ to (16) ~~is~~

$$\frac{d}{dt}(\ell+\Omega_\zeta) = \frac{\partial}{\partial t}(\ell+\Omega_\zeta) + u_g\frac{\partial}{\partial\xi}(\ell+\Omega_\zeta) + v_g\frac{\partial}{\partial\eta}(\ell+\Omega_\zeta) + \tilde{w}\frac{\partial}{\partial\zeta}(\ell+\Omega_\zeta)$$

In order to transform the rest of the terms of (17) to the new variables, let us introduce the temporary designations

$$u(t,x,y,\zeta) = U(t,\xi,\eta,\zeta); \quad v(t,x,y,\zeta) = V(t,\xi,\eta,\zeta)$$

$$\tilde{w}(t,x,y,\zeta) = W(t,\xi,\eta,\zeta)$$

Then we have

(18)

$$\frac{\partial\tilde{w}}{\partial\zeta} = \frac{\partial W}{\partial\zeta} + \frac{\partial W}{\partial\xi}\frac{1}{\ell}\frac{\partial v}{\partial\zeta} - \frac{\partial W}{\partial\eta}\frac{1}{\ell}\frac{\partial u}{\partial\zeta}$$

$$\frac{\partial\tilde{w}}{\partial x} = \frac{\partial W}{\partial\xi}\left(1+\frac{1}{\ell}\frac{\partial v}{\partial x}\right) - \frac{\partial W}{\partial\eta}\frac{1}{\ell}\frac{\partial u}{\partial x}$$

$$\frac{\partial\tilde{w}}{\partial y} = \frac{\partial W}{\partial\eta}\left(1-\frac{1}{\ell}\frac{\partial u}{\partial y}\right) + \frac{\partial W}{\partial\xi}\frac{1}{\ell}\frac{\partial v}{\partial y}$$

Further it follows to express, with the aid of the same such relation all the derivatives of u and v in the new variables and put them into equation (18). However, since for calculations we shall neglect all quantities of the order ε^2 in comparison with unity, the calculations should be essentially simplified. As $\frac{1}{\ell}\frac{\partial v}{\partial x}$, $\frac{1}{\ell}\frac{\partial u}{\partial x}$ etc., a considered themselves as quantities of the order ε, we can rewrite the latter equation (18) thus:

$$\frac{\partial \tilde{w}}{\partial x} = \frac{\partial W}{\partial \xi} + \varepsilon\left(\frac{\partial \tilde{w}}{\partial s}\right); \quad \frac{\partial \tilde{w}}{\partial y} = \frac{\partial W}{\partial \eta} + \varepsilon\left(\frac{\partial \tilde{w}}{\partial s}\right).$$

Here $\varepsilon\left(\frac{\partial \tilde{w}}{\partial s}\right)$ denotes the quantity of the order ε relati to the horizontal derivative of the quantity $\tilde{w}$.

Then we have

$$(19) \qquad -\ell\frac{\partial \tilde{w}}{\partial \zeta} + \frac{\partial \tilde{w}}{\partial x}\frac{\partial v}{\partial \zeta} - \frac{\partial \tilde{w}}{\partial y}\frac{\partial u}{\partial \zeta} = \ell\frac{\partial W}{\partial \zeta} + \varepsilon\left(\frac{\partial \tilde{w}}{\partial s}\frac{\partial V_s}{\partial \zeta}\right)$$

The meaning of the designation $\varepsilon\left(\frac{\partial \tilde{w}}{\partial s}\frac{\partial V_s}{\partial \zeta}\right)$ is clear from th preceeding. Since the term in brackets has the order ε relative to the chief terms of equation (17), then with an accuracy to a value of the order ε^2 we can replace the left side of (19) by $\ell\frac{\partial W}{\partial \zeta}$. Turni then to the entry in (17) the quantity $\Omega_1\frac{\partial \tilde{W}}{\partial \zeta}$, we are easily convinced, that in this case for the expression $\partial \tilde{w}/\partial s$ in the new variables still less accuracy is needed and it is possible to write

$$\Omega_1 \frac{\partial \tilde{W}}{\partial \zeta} \approx \Omega_\zeta \frac{\partial W}{\partial \zeta}$$

Collecting all the results obtained, we bring equation (17) to the form

$$\text{(20)} \quad \frac{\partial(l+\Omega_1)}{\partial t} + u_g \frac{\partial}{\partial \xi}(l+\Omega_1) + v_g \frac{\partial}{\partial \eta}(l+\Omega_1) + \tilde{W}\frac{\partial}{\partial \zeta}(l+\Omega_\zeta) - (l+\Omega_i)\frac{\partial \tilde{W}}{\partial \zeta} = 0$$

Here the old designation $W = \tilde{W}$ has been restored, but it follows to remember, that now $\tilde{W}$ is presented as a function of t, ξ, η, ζ. Let us transform equation (8) to the new variables. We have (ϕ in the new variables we shall designate by $\hat{\phi}$)

$$\text{(21)} \quad \theta = -\frac{1}{R}\zeta^{1-\varkappa}\left(\frac{\partial \hat{\phi}}{\partial \zeta} + \frac{\partial \hat{\phi}}{\partial \xi}\frac{1}{l}\frac{\partial v}{\partial \zeta} - \frac{\partial \hat{\phi}}{\partial \eta}\frac{1}{l}\frac{\partial u}{\partial \zeta}\right).$$

The geostrophic relations (10) in the new variables take the form

$$\text{(22)} \quad \begin{aligned} u_g &= -\frac{1}{l}\frac{\partial \hat{\phi}}{\partial \eta}\left(1 - \frac{1}{l}\frac{\partial u}{\partial y}\right) - \frac{1}{l}\frac{\partial \hat{\phi}}{\partial \xi}\frac{1}{l}\frac{\partial v}{\partial y} \\ v_g &= \frac{1}{l}\frac{\partial \hat{\phi}}{\partial \xi}\left(1 + \frac{1}{l}\frac{\partial v}{\partial x}\right) - \frac{1}{l}\frac{\partial \hat{\phi}}{\partial \eta}\frac{\partial u}{\partial x} \end{aligned}$$

Then with accuracy to the second order of smallness we can persent (22) in the form

$$\text{(23)} \quad u_g = -\frac{1}{l}\frac{\partial\left(\hat{\phi} + \frac{u^2+v^2}{2}\right)}{\partial \eta}; \quad v_g = \frac{1}{l}\frac{\partial\left(\hat{\phi} + \frac{u^2+v^2}{2}\right)}{\partial \xi}$$

It is not difficult to verify, that carrying out the same such transformations in (21), we obtain the quite accurate expression

$$(24) \qquad \theta = -\frac{1}{R} \zeta^{1-\lambda} \frac{\partial}{\partial \zeta} \left(\hat{\phi} + \frac{u^2+v^2}{2} \right)$$

If we find $\partial\theta/\partial\xi$ or $\partial\theta/\partial\eta$ from (24), then the error will have the second order of smallness.

Now let us differentiate equation (16) with respect to ζ. The on the strength of (23) and (24) the quantity

$$\frac{\partial u_g}{\partial \zeta}\frac{\partial \theta}{\partial \xi} + \frac{\partial v_g}{\partial \zeta}\frac{\partial \theta}{\partial \eta}$$

is equal to zero, and we find

$$(25) \qquad \left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial \xi} + v_g \frac{\partial}{\partial \eta} + \tilde{w}\frac{\partial}{\partial \zeta} \right) \frac{\partial \theta}{\partial \zeta} + \frac{\partial \tilde{w}}{\partial \zeta}\frac{\partial \theta}{\partial \zeta} = 0$$

From (20) and (25) we obtain

$$(26) \left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial \xi} + v_g \frac{\partial}{\partial \eta} + \tilde{w}\frac{\partial}{\partial \zeta} \right) \left[\ln (l + \Omega_\zeta) + \ln \frac{\partial \theta}{\partial \zeta} \right] =$$

So, again we obtained the invariant $(l + \Omega_\zeta)\frac{\partial \theta}{\partial \zeta}$ but in the variables t, ξ, η, ζ and for the replacement of the actual velocities u, v by the geostrophic relations. The characteristic of the conservation of this quantity is fulfilled with considerable accuracy.

From the deduction it follows, that this accuracy, under normal conditions when the flux of heat and the amount of turbulent transfer of the motion in the free atmosphere is small, is completely sufficient for all kinds of practical calculations.

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